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MAHLER MEASURE OF THE COLORED JONES POLYNOMIAL AND THE VOLUME CONJECTURE

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ABSTRACT. In this note, I will discuss a possible relation between the Mahler measure of the colored Jones polynomial and the volume conjecture. In particular, I will study the colored Jones polynomial of the figure-eight knot on the unit circle. I will also propose a method to prove the volume conjecture for satellites of the figure-eight knot.

1. MAHLER MEASURE

Let $f(t)$ be a (non-zero) Laurent polynomial in t with coefficient in $\mathbb{Z}$. The *Mahler measure* $\mathbf{M}(f)$ of f [5, 6, 14] is defined to be

$$\mathbf{M}(f) := \exp\left(\int_0^1 \log\left|f\left(\exp(2\pi\sqrt{-1}x)\right)\right| dx\right)$$

It is known that $\mathbf{M}(f)$ is the product of the absolute values of the leading coefficient and all the roots that are greater than one. It is convenient to define its logarithmic version:

$$\mathbf{m}(f) := \int_0^1 \log\left|f\left(\exp(2\pi\sqrt{-1}x)\right)\right| dx.$$

Then the logarithmic Mahler measure can be regarded as a sort of 'mean' of the logarithms of the values on the unit circle. Visit the web pages

`http://mathworld.wolfram.com/MahlerMeasure.html`

for more about the Mahler measure and also

`http://math.ucr.edu/~xl/knotprob/knotprob.html`

for problems on the Mahler measure of the Jones polynomial.

2. MAHLER MEASURE OF THE ALEXANDER POLYNOMIAL

Let K be a knot in the three-sphere S^3 and $M_N(K)$ be the N-fold cyclic branched covering over S^3 branched along K. Then it is well known that the order of the first homology group of $M_N(K)$ can be obtained in terms of the Alexander polynomial $\Delta(K;t)$ of K (see for example [4, Corollary 9.8]).

Theorem 2.1.

$$|H_1(M_N(K);\mathbb{Z})| = \prod_{d=1}^{N-1} \Delta(K, \exp(2d\pi\sqrt{-1}/N)), \tag{2.1}$$

where $|A|$ denotes the cardinality of a set A if A is a finite set and 0 if it is infinite.

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If we take the logarithm of the both side of Equation (2.1) and divide by N, we have

$$\frac{\log|H_1(M_N(K);\mathbb{Z})|}{N}=\frac{\sum_{d=1}^{N-1}\log\left|\Delta(K;\exp(2\pi d\sqrt{-1}/N))\right|}{N}$$

When N grows, the right hand side approaches to the 'mean' of the logarithms of the values of $\Delta(K;t)$ on the unit circle, the logarithmic Mahler measure. In fact the following theorem is known to be true.

Theorem 2.2 (D. Silver and S. Williams [15]).

$$\lim_{N\to\infty}\frac{\log|H_1(M_N(K);\mathbb{Z})|}{N}=\mathbf{m}(\Delta(K;t))$$

See [2, 1, 13] for other topics of the homology of the branched cyclic cover over a knot. See also [16] for the Mahler measure of the Alexander polynomial of a link.

3. Mahler measure of the colored Jones polynomials

Let $J_N(K;t)$ be the N-dimensional colored Jones polynomial of a knot K normalized so that $J_N(O;t)=1$ for the unknot O. We want to know the asymptotic behavior of $J_N(K;t)$ for large N.

Since

$$\begin{aligned}\mathbf{m}\left(J_N(K;t)\right)&=\int_0^1\log\left|J_N\left(K;\exp(2\pi\sqrt{-1}x)\right)\right|dx\\&=\int_0^N\frac{\log\left|J_N\left(K;\exp(2r\pi\sqrt{-1}/N)\right)\right|}{N}dr,\end{aligned}$$

it is helpful to study the asymptotic behavior of $\log\left|J_N\left(K;\exp(2\pi\sqrt{-1}r/N)\right)\right|$ for a fixed r. Note that for $r=1$, this problem is nothing but the volume conjecture [11, 8, 10, 9, 19, 18, 17, 12].

In the following sections I will discuss the colored Jones polynomials of the figure-eight knot evaluated on the unit circle.

4. Some calculations about the figure-eight knot

Let E denote the figure-eight knot 4_1. Due to K. Habiro and T. Le, the following formula is known.

$$J_N(E;t)=\sum_{k=0}^{N-1}\prod_{j=1}^{k}\left(t^{(N+j)/2}-t^{-(N+j)/2}\right)\left(t^{(N-j)/2}-t^{-(N-j)/2}\right). \tag{4.1}$$

Using this formula we can prove the following result.

Theorem 4.1. *Let r be a positive integer or a real number satisfying $5/6<r<7/6$. Then*

$$\lim_{N\to\infty}2\pi\frac{\log\left|J_N\left(E;\exp(2r\pi\sqrt{-1})\right)\right|}{N}=\frac{2\Lambda(r\pi+\theta(r)/2)-2\Lambda(r\pi-\theta(r)/2)}{r},$$

where $\Lambda(z):=-\int_0^z\log|\sin x|dx$ is the Lobachevski function and $\theta(r)$ is the smallest positive number satisfying $\cos\theta(r)=\cos(2r\pi)-1/2$.

In particular, if r is a positive integer then

$$2\pi\lim_{N\to\infty}\frac{\log\left|J_N\left(E;\exp(2r\pi\sqrt{-1}/N)\right)\right|}{N}=\frac{\mathrm{Vol}(S^3\setminus E)}{r}.$$

Proof of Theorem 4.1 when r is a positive integer. Replacing t with $\exp(2r\pi\sqrt{-1}/N)$ in Equation (4.1), we have

$$J_N(E;\exp(2r\pi\sqrt{-1}/N)) = \sum_{k=0}^{N-1}\prod_{j=1}^{k}\{2\sin(jr\pi/N)\}^2$$

If we put $f(k) := \prod_{j=1}^{k}\{2\sin(jr\pi/N)\}^2$, then f takes its maximum at $kr\pi/N = 5\pi/6$ if N is large. Therefore

$$\begin{aligned}\lim_{N\to\infty}\frac{\log\left|J_N\left(E;\exp(2r\pi\sqrt{-1}/N)\right)\right|}{N} &= 2\lim_{N\to\infty}\sum_{j=1}^{5N/6r}\frac{\log\left(2\sin(jr\pi/N)\right)}{N}\\ &= \frac{2}{r\pi}\lim_{N\to\infty}\int_0^{5\pi/6}\log\left(2\sin x\right)\,dx\\ &= -\frac{2}{r\pi}\Lambda(5\pi/6)\\ &= \frac{\operatorname{Vol}(S^3\setminus E)}{2r\pi}.\end{aligned}$$

See [8, Theorem 4.2] for details. □

Remark 4.2. The case where $r = 1$ is due to R. Kashaev [3] and T. Ekholm [8].

Proof of Theorem 4.1 when $5/6 < r < 1$. We will assume N is sufficiently large so that j/N can behave as if it is a continuous parameter.

Put $\omega := \exp(2\pi\sqrt{-1}/N)$. Since

$$\omega^{r(N+j)/2} - \omega^{-r(N+j)/2} = 2\sqrt{-1}\sin(r(N+j)\pi/N)$$

and

$$\omega^{r(N-j)/2} - \omega^{-r(N-j)/2} = 2\sqrt{-1}\sin(r(N-j)\pi/N),$$

we have

$$\begin{aligned}&\prod_{j=1}^{k}\left(\omega^{r(N+j)/2} - \omega^{-r(N+j)/2}\right)\left(\omega^{r(N-j)/2} - \omega^{-r(N-j)/2}\right)\\ &\qquad = \prod_{j=1}^{k}4\sin(rj\pi/N + r\pi)\sin(rj\pi/N - r\pi).\end{aligned}$$

Put

$$\begin{aligned}g(j) :&= 4\sin(rj\pi/N + r\pi)\sin(rj\pi/N - r\pi)\\ &= 2\cos(2r\pi) - 2\cos(2rj\pi/N)\end{aligned}$$

and

$$f(k) := \prod_{j=1}^{k} g(j)$$

so that $J_N(E;\omega^r) = \sum_{k=0}^{N-1} f(k)$. We also put

$$A := \frac{N(1-r)}{r},\quad B := \frac{N\theta(r)}{2r\pi},\quad \text{and}\quad C := \frac{N(2\pi - \theta(r))}{2r\pi}$$

where $\theta(r)$ is the smallest positive number satisfying $\cos\theta(r) = \cos(2r\pi) - 1/2$ as before. Note that since $5/6 < r < 1$, $1/2 < \cos(2r\pi) < 1$ and so the equation $\cos\theta(r) = \cos(2r\pi) - 1/2$ has a solution.

Note that $0 < A < B < C < N$ (see Figure 1).

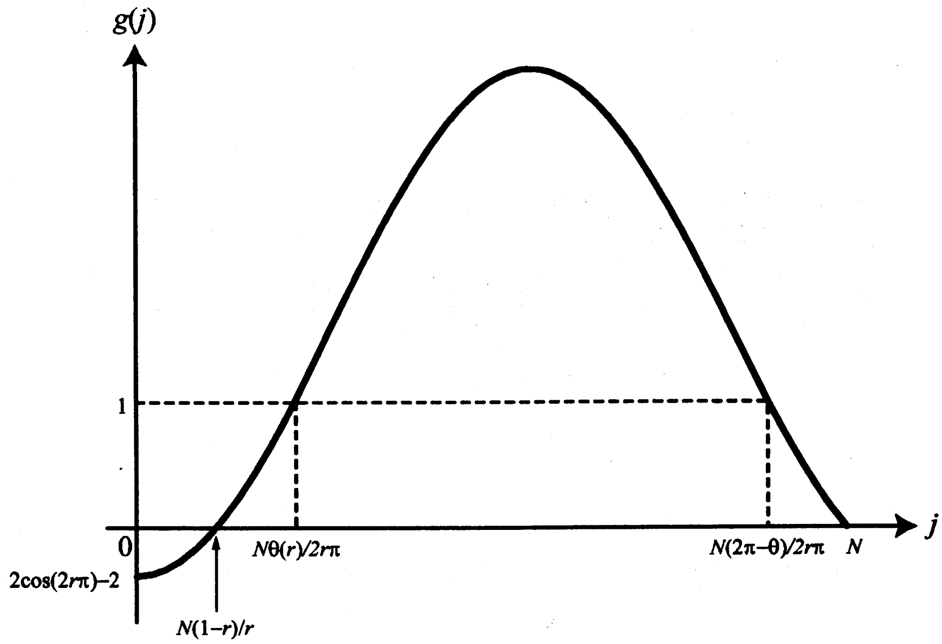

FIGURE 1. Graph of $g(j)$ when $5/6 < r < 1$

Since we have

(1) $g(j) < 0$ for $j < A$, and $g(j) > 0$ for $j > A$, and
(2) $f_j > 1$ for $B < j < C$,

we see

(3) If $j < A$ then the signs of $f(j)$ alternate, that is, $f(j-1)f(j) < 0$, and if $j > A$ then the signs of $f(j)$ are constant, and
(4) $|f(0)| > |f(1)| > \cdots > |f(B)|$ and $|f(B+1)| < \cdots < |f(C)|$.

Let f_{MAX} [1] be the maximum of $\{|f_j|\}$ for $0 \le j \le N-1$. Note that $f_{\mathrm{MAX}} = f(C)$. We can show the following inequality.

Claim 4.3.

$$0 < f_{MAX} - 1 \le |J_N(E;\omega^r)| \le N\, f_{MAX}$$

Proof of the Claim 4.3. We only show the second inequality for the case where A is even. In this case since $f(0) = 1$, $f(2j-1) + f(2j) < 0$ for $2j < A$, and $f(j) < 0$ for $j \ge A-1$, we have

$$\begin{aligned}
&|J_N(E;\omega^r)| \\
&\quad = \big|f(0) + \{f(1)+f(2)\} + \{f(3)+f(4)\} + \cdots + \{f(A-3)+f(A-2)\} \\
&\qquad + f(A-1) + f(A) + f(A+1) + \cdots + f(N-1)\big| \\
&\quad = |f(1)+f(2)| + |f(3)+f(4)| + \cdots + |f(A-3)+f(A-2)| \\
&\qquad + |f(A-1)| + |f(A)| + \cdots + |f(N-1)| \\
&\qquad - 1 \\
&\quad > f_{\mathrm{MAX}} - 1
\end{aligned}$$

and the second equality follows. □

[1]MAX are temporarily Nana, Reina, and Lina.

Therefore we have

$$
\begin{aligned}
&\lim_{N\to\infty} \frac{\log|J_N(E;\omega^r)|}{N} \\
&= \lim_{N\to\infty} \frac{\log(f_{\text{MAX}})}{N} \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{j=0}^{C}\left\{\log\left(2\sin\left(\frac{rj\pi}{N}+r\pi-\pi\right)\right)+\log\left(2\sin\left(-\frac{rj\pi}{N}+r\pi\right)\right)\right\} \\
&= \frac{1}{r\pi}\int_{r\pi-\pi}^{r\pi-\theta(r)/2}\log(2\sin x)dx + \frac{1}{r\pi}\int_{r\pi-\pi+\theta(r)/2}^{r\pi}\log(2\sin x)dx \\
&= \frac{1}{r\pi}\left(\Lambda(r\pi-\pi)-\Lambda(r\pi-\theta(r)/2)+\Lambda(r\pi-\pi+\theta(r)/2)-\Lambda(r\pi)\right) \\
&= \frac{1}{r\pi}\left(\Lambda(r\pi+\theta(r)/2)-\Lambda(r\pi-\theta(r)/2)\right).
\end{aligned}
$$

Here we use the π-periodicity of the Lobachevski function. (See [7].) □

Proof when $1 < r < 7/6$. The proof is similar to the case where $5/6 < r < 1$. See Figure 2. □

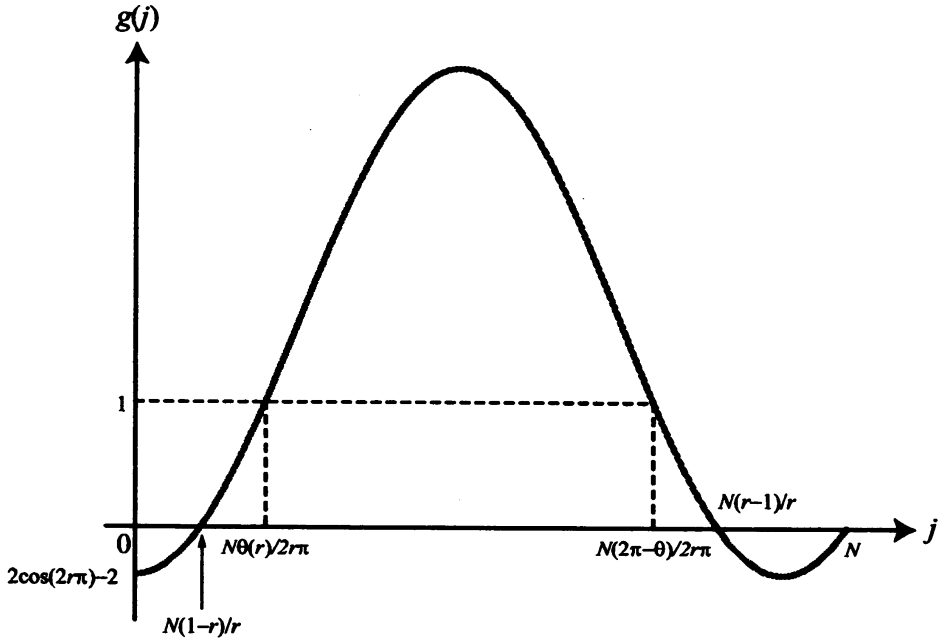

FIGURE 2. Graph of $g(j)$ when $1 < r < 7/6$

As a corollary we have

Corollary 4.4.

$$
\lim_{r\to 1}\left\{\lim_{N\to\infty}\frac{\log|J_N(E;\exp(2r\pi\sqrt{-1}))|}{N}\right\} = \lim_{N\to\infty}\frac{\log|J_N(E;\exp(2\pi\sqrt{-1}))|}{N}
$$

By some calculation using PARI-GP [2] and MAPLE V, it seems that the following equality holds.

$$2r\pi \lim_{N\to\infty} \frac{\log|J_N(E;\omega^r)|}{N} = \begin{cases} V(r) & \text{if } 0 \le r \le 1, \\ W(r-[r]) & \text{if } r > 1, \end{cases} \tag{4.2}$$

where $[r]$ denotes the greatest integer which does not exceed r, and

$$V(x) := \begin{cases} 0 & \text{if } 0 \le x < 1/6, \\ \Lambda(x\pi + \theta(x)/2 - \pi/2) - \Lambda(x\pi - \theta(x)/2 - \pi/2) & \text{if } 1/6 \le x < 3/4, \\ \Lambda(x\pi + \theta(x)/2) - \Lambda(x\pi - \theta(x)/2) & \text{if } 3/4 \le x \le 1, \end{cases}$$

and

$$W(x) := \begin{cases} \Lambda(x\pi) + \theta(x)/2 - \Lambda(x\pi - \theta(x)/2) & \text{if } 0 \le x < 1/4, \\ \Lambda(x\pi) + \theta(x)/2 - \pi/2) - \Lambda(x\pi - \theta(x)/2 - \pi/2) & \text{if } 1/4 \le x < 3/4, \\ \Lambda(x\pi + \theta(x)/2) - \Lambda(x\pi - \theta(x)/2) & \text{if } 3/4 \le x \le 1. \end{cases}$$

See Figures 3 and 4 for graphs of V and W. See also Figures 6, 7, 8, 9, 10, and 11

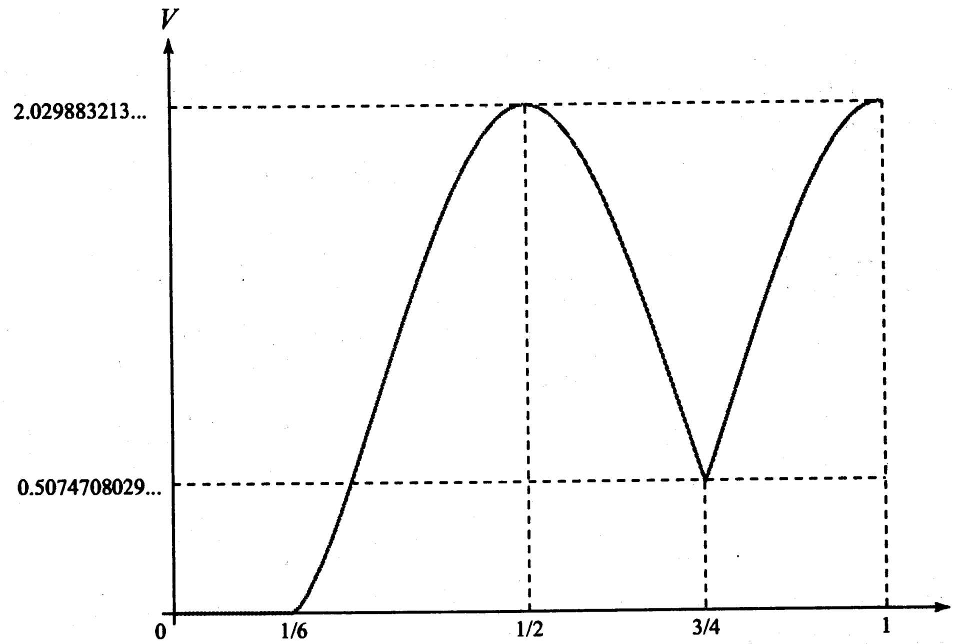

FIGURE 3. Graph of V, where 2.029883213... is the volume of the figure-eight knot complement.

for some results of calculations supporting Equation 4.2.

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2 GP/PARI CALCULATOR Version 2.0.20 (beta)
i586 running cygwin_98-4.10 (ix86 kernel) 32-bit version
(readline v1.0 enabled, extended help not available)

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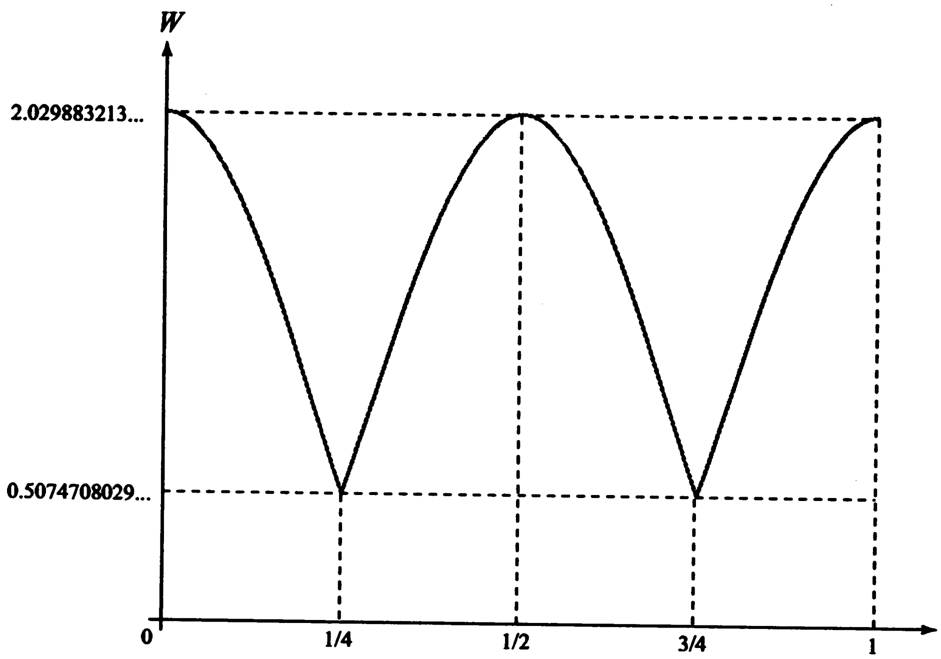

FIGURE 4. Graph of W.

If Equation (4.2) is true, one could have the following result on the asymptotic behavior of the logarithmic Mahler measure of the colored Jones polynomials of the figure-eight knot.

Remark 4.5. Caution! There are *fake* calculations in the following.

$$\begin{aligned}
\lim_{N\to\infty}\frac{\mathrm{m}\,(J_N(E;t))}{\log N} &= \lim_{N\to\infty}\frac{1}{\log N}\int_0^1 \log\left|J_N(E;\exp(2\pi\sqrt{-1}x))\right| dx \\
&\underset{?}{=} \lim_{N\to\infty}\frac{1}{\log N}\int_0^N \frac{\log\left|J_N(E;\exp(2\pi\sqrt{-1}r/N))\right|}{N}dr \\
&\underset{?}{=} \lim_{N\to\infty}\frac{1}{2\pi\log N}\left\{\int_0^1\frac{V(r)}{r}dr+\sum_{k=1}^{N-1}\int_k^{k+1}\frac{W(r-[r])}{r}dr\right\} \\
&= \lim_{N\to\infty}\frac{1}{2\pi\log N}\left\{\int_0^1\frac{V(r)}{r}dr+\sum_{k=1}^{N-1}\int_0^1\frac{W(r)}{r+k}dr\right\},
\end{aligned}$$

where $\underset{?}{=}$ means that there is a doubt in the equality. At the first I use N in the integral, which should be independent of N, and at the second I assume (4.2).

Now since

$$\frac{1}{k+1}\le\frac{1}{r+k}\le\frac{1}{k}$$

for $0\le r\le 1$, we have

$$\int_0^1\frac{W(r)}{k+1}dr\le\int_0^1\frac{W(r)}{r+k}dr\le\int_0^1\frac{W(r)}{k}dr.$$

Therefore we have

$$\sum_{k=1}^{N-1}\int_0^1\frac{W(r)}{k+1}dr\le\sum_{k=1}^{N-1}\int_0^1\frac{W(r)}{r+k}dr\le\sum_{k=1}^{N-1}\int_0^1\frac{W(r)}{k}dr.$$

Since

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N-1}\frac{1}{k+1}}{\log N}=\lim_{N\to\infty}\frac{\sum_{k=1}^{N-1}\frac{1}{k}}{\log N}=1$$

and $V(r)=0$ for $0\le r\le 1/6$, we finally have

$$2\pi\lim_{N\to\infty}\frac{\mathbf{m}\left(J_N(E;t)\right)}{\log N}\underset{?}{=}\int_0^1 W(r)dr=1.450191516....$$

5. Satellites of the figure-eight knot

In this section, I would like to study the volume conjecture for the $(2,1)$-cable and the Whitehead double of the figure-eight knot. Linear skein method gives us formulas to describe the colored Jones polynomials of such knots but one of the difficulties is that the value of the unknot is not 1 but $(t^{N/2}-t^{-N/2})/(t^{1/2}-t^{-1/2})$ (see for example [4, Chapter 14]), and so they vanish if we evaluate them at the N-th root of unity. To avoid this I will use Corollary 4.4 to analyze the asymptotic behaviors of the colored Jones polynomials. Unfortunately, I cannot give a rigorous result here but I hope that this method gives an insight to solve the volume conjecture for satellite knots.

Remark 5.1. Caution! There are many *fake* arguments in this section.

Let E^2 be the $(2,1)$-cable of the figure-eight knot. By using techniques in [4, Chapter 14], we see

$$J_N(E^2;t)(t^{N/2}-t^{-N/2})/(t^{1/2}-t^{-1/2})=\sum_{\substack{c:\text{ odd}\\ c\le 2N-1}}u(c;t^{1/4})J_c(E;t),$$

where $u(c;t^{1/4})$ is a monomial in $t^{1/4}$. Replacing t with ω^r with $5/6<r<7/6$ $(r\ne 1)$, we have

$$J_N(E^2;\omega^r)=\frac{\sin(r\pi/N)}{\sin(r\pi)}\sum_{\substack{c:\text{ odd}\\ 1\le c\le 2N-1}}u(c;\omega^{r/4})J_c(E;\omega^r).$$

Note that $\sin(r\pi)\ne 0$. If one could show that the maximum of the terms in the summation dominates the limit, which is a kind of saddle point method, we could have

$$\begin{aligned}\lim_{N\to\infty}\frac{\log\left|J_N(E^2;\omega)\right|}{N}&\underset{?}{=}\lim_{r\to 1}\left\{\lim_{N\to\infty}\frac{\log\left|J_N(E^2;\omega^r)\right|}{N}\right\}\\&\underset{?}{=}\lim_{r\to 1}\left\{\frac{\log\left|\max_{1\le c\le 2N-1}J_c(E,\omega^r)\right|}{N}\right\}\\&\underset{?}{=}\lim_{r\to 1}\left\{\frac{\log\left|J_N(E,\omega^r)\right|}{N}\right\}\\&=\frac{\log\left|J_N(E,\omega)\right|}{N},\end{aligned}$$

proving the volume conjecture for the $(2,1)$-cable of the figure-eight knot. Here $\underset{?}{=}$ indicates that there is a doubt in the equality; at the fist equality, I change the order of the limits, at the second, I assume the maximum dominates the limit, and at the third, I assume that $J_c(E,\omega^r)$ takes its maximum at $c=N$, which can be observed by calculation using PARI. See Figure 5.

I believe that the gaps here are not so big.

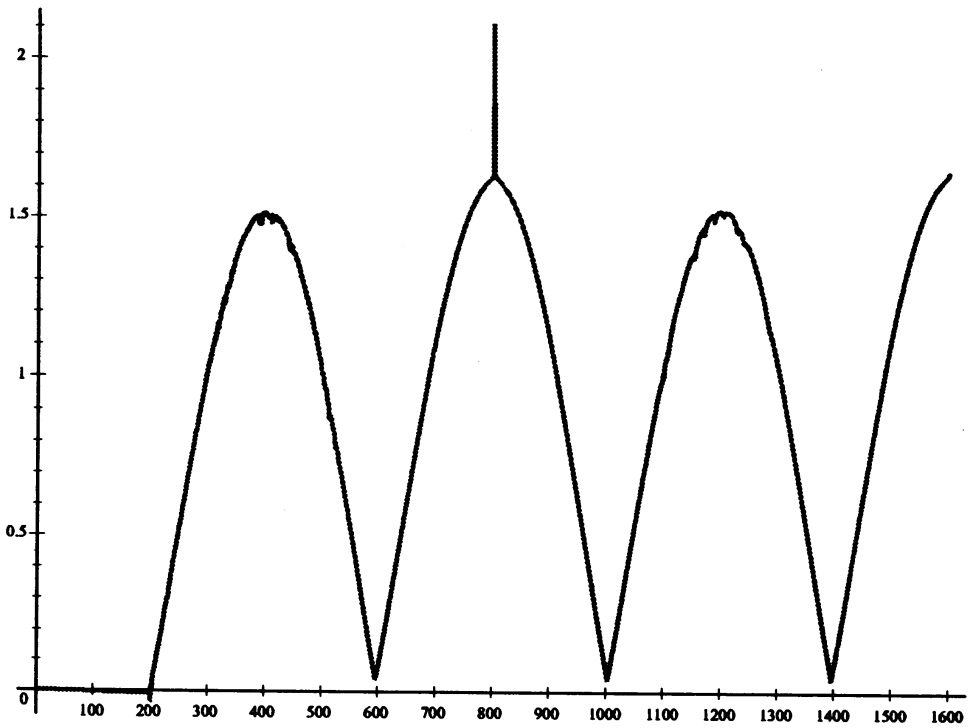

FIGURE 5. Plot of $(c, 2\pi \log |J_c(E;\omega)|/N)$ with $N = 800$.

Let $D(E)$ be the Whitehead double of the figure-eight knot (with any framing). Then using similar techniques we have

$$J_N(D(E);t)(t^{N/2} - t^{-N/2})/(t^{1/2} - t^{-1/2}) = \sum_{\substack{c:\text{ odd}\\ c\leq 2N-1}} v(c;t) J_c(E;t),$$

with

$$v(c;t) = \sum_{\substack{d:\text{ odd}\\ d\leq 2N-1}} \frac{\Delta_d\, \theta(N-1, N-1, c-1)}{\Delta_c\, \theta(N-1, N-1, d-1)} \begin{Bmatrix} N-1 & N-1 & c \\ N-1 & N-1 & d \end{Bmatrix}$$

where $\Delta_x, \theta(x,y,z)$ and $\begin{Bmatrix} x & y & z \\ u & v & w \end{Bmatrix}$ are defined in [4, Chapter 14]. Similar calculation shows that for the Whitehead link W we have

$$J_N(W;t)(t^{N/2} - t^{-N/2})/(t^{1/2} - t^{-1/2}) = \sum_{\substack{c:\text{ odd}\\ c\leq 2N-1}} v(c;t) J_{N,c}(H;t),$$

where $J_{N,c}(H;t)$ is the colored Jones polynomial of the Hopf link H colored with N and c, which is equal to $\Delta_{(N-1)(c-1)}$.

Now we have the following *fake* calculations with doubtful equalities:

$$\lim_{N\to\infty} \frac{\log |J_N(D(E);\omega)|}{N} \underset{?}{=} \lim_{r\to 1} \left\{ \lim_{N\to\infty} \frac{\log |J_N(D(E);\omega^r)|}{N} \right\}$$

$$\underset{?}{=} \lim_{r\to 1} \left\{ \frac{\log \left| \max_{1\leq c\leq 2N-1} v(c;\omega^r) J_c(E,\omega^r) \right|}{N} \right\}$$

$$\begin{aligned}
&\underset{?}{=} \lim_{r\to 1}\left\{\frac{\log|v(N;\omega^r)J_N(E;\omega^r)|}{N}\right\} \\
&= \lim_{r\to 1}\frac{\log|v(N;\omega^r)|}{N} + \lim_{r\to 1}\frac{\log|J_N(e;\omega^r)|}{N} \\
&= \lim_{r\to 1}\frac{\log|v(N;\omega^r)|}{N} + \frac{\log|J_N(E,\omega)|}{N},
\end{aligned}$$

On the other hand

$$\begin{aligned}
\lim_{N\to\infty}\frac{\log|J_N(W;\omega)|}{N} &\underset{?}{=} \lim_{r\to 1}\left\{\lim_{N\to\infty}\frac{\log|J_N(W;\omega^r)|}{N}\right\} \\
&\underset{?}{=} \lim_{r\to 1}\left\{\frac{\log|v(N;\omega^r)J_{N,N}(H;\omega^r)|}{N}\right\} \\
&= \lim_{r\to 1}\frac{\log|v(N;\omega^r)|}{N}
\end{aligned}$$

since $J_{N,N}(W,\omega^r)$ can be expressed in terms of sine of $1/N$. Therefore if we accept these calculations, we could prove

$$\lim_{n\to\infty}\frac{\log|J_N(D(E),\omega)|}{N} = \lim_{n\to\infty}\frac{\log|J_{N,N}(W,\omega)|}{N} + \lim_{n\to\infty}\frac{\log|J_N(E,\omega)|}{N}.$$

Noting that the complement of $D(E)$ is the union of those of the figure-eight knot and the Whitehead link, which is the volume conjecture for the Whitehead double of the figure-eight knot.

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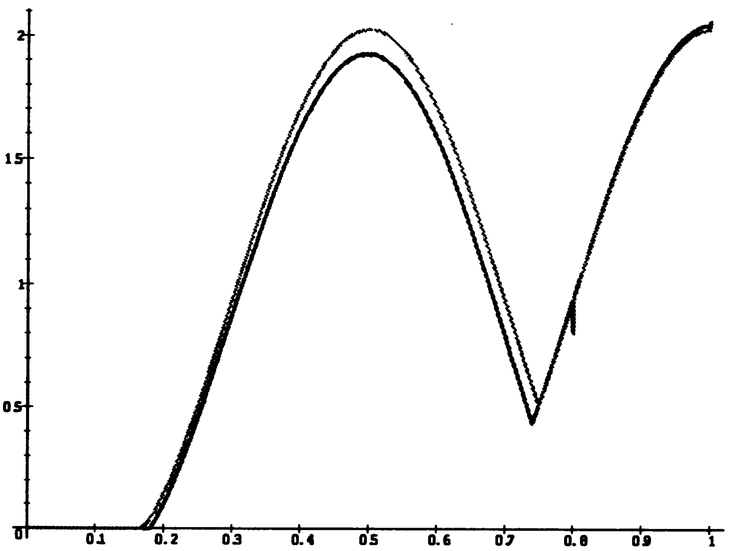

FIGURE 6. Graph of W (gray) and $2r\pi \log |J_N(E;\omega^r)|/N$ with $N = 2000$ (black) for $0 \le r \le 1$.

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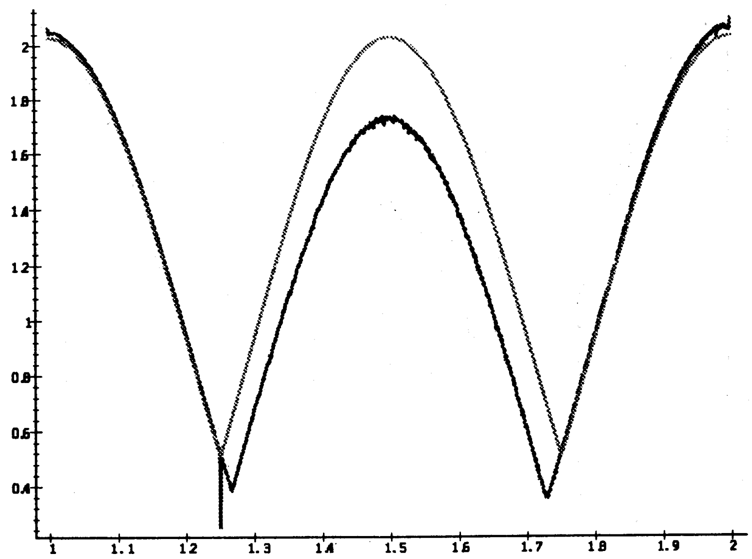

FIGURE 7. Graph of W (gray) and $2r\pi \log |J_N(E;\omega^r)|/N$ with $N = 2000$ (black) for $1 \le r \le 2$.

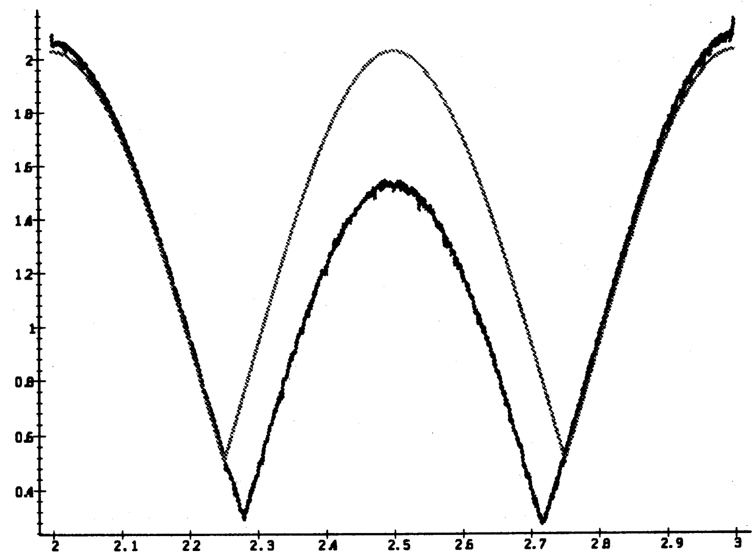

FIGURE 8. Graph of W (gray) and $2r\pi \log |J_N(E;\omega^r)|/N$ with $N = 2000$ (black) for $2 \le r \le 3$.

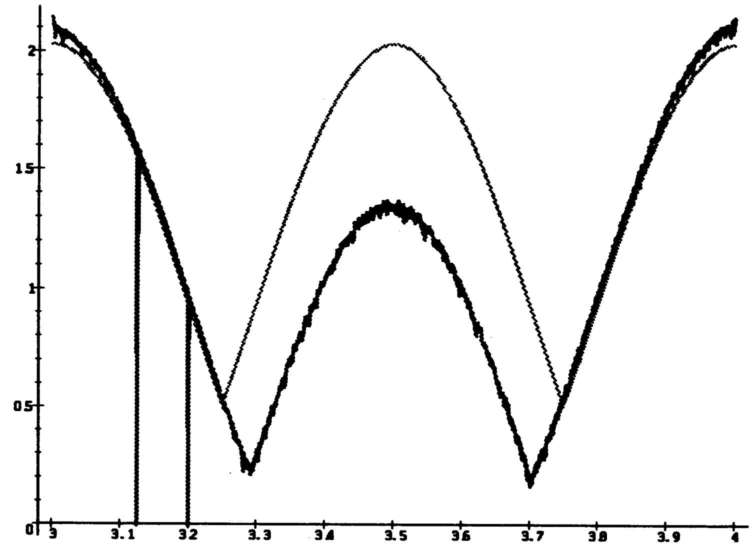

FIGURE 9. Graph of W (gray) and $2r\pi \log |J_N(E;\omega^r)|/N$ with $N = 2000$ (black) for $3 \le r \le 4$.

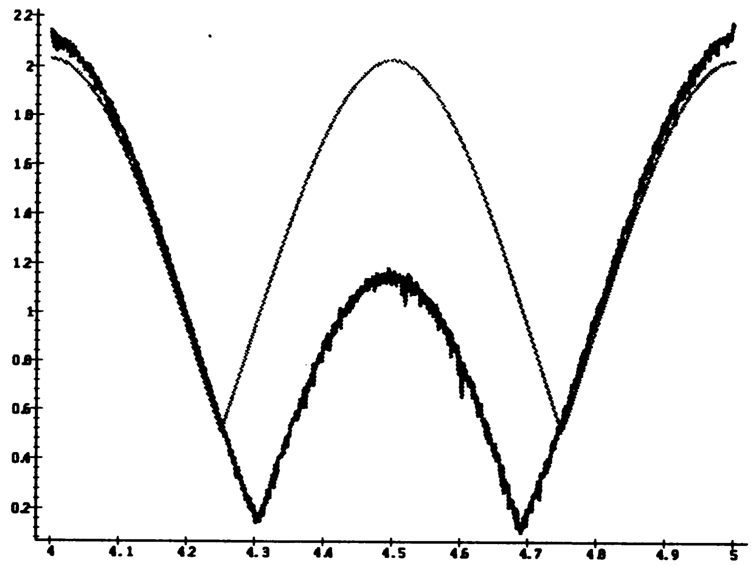

FIGURE 10. Graph of W (gray) and $2r\pi \log |J_N(E;\omega^r)|/N$ with $N = 2000$ (black) for $4 \le r \le 5$.

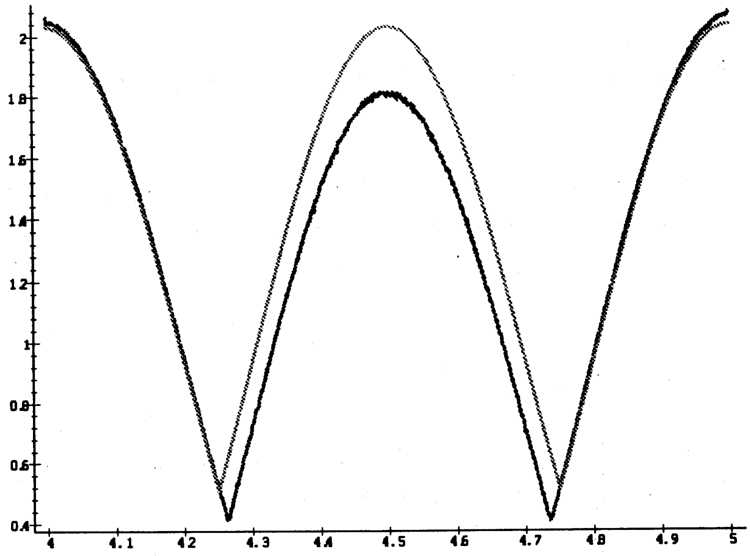

FIGURE 11. Graph of W (gray) and $2r\pi \log |J_N(E;\omega^r)|/N$ with $N = 8000$ (black) for $4 \leq r \leq 5$.